Gravity in one dimension: The critical population

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The failure of a one-dimensional gravitational system to relax to equilibrium on predicted time scales has raised questions concerning the ergodic properties of the dynamics. A failure to approach equilibrium could be caused by the segmentation of the phase space into isolated regions from which the system cannot escape. In general, each region may have distinct ergodic properties. By numerically investigating the stability of two classes of periodic orbits for the N-body system in a previous work [Phys. Rev. A **46**, 837 (1992)], we demonstrated that phase-space segmentation occurred when $N \leq 10$. Tentative results suggested that segmentation also occurred for $11 \leq N \leq 20$. Here this work has been refined. Based on calculations of Lyapunov characteristic numbers, we argue that segmentation disappears and the system is both ergodic and mixing for $N \geq 11$, the critical population.

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I. INTRODUCTION

The system of infinitely extended parallel mass sheets moving solely under their mutual gravitational attraction has provided a model system for testing numerous theories of dynamical relaxation. Members of this one-dimensional gravitating system (one-dimensional galaxy) are not subject to either escape or a singular "interparticle" interaction, so the system is especially amenable to accurate computer simulation. Early studies showed that systems with large populations failed to relax to equilibrium on predicted time scales and thus raised questions concerning the ergodic properties of the motion. Some time ago we showed that, for small systems (three or four sheets), the phase space has coexisting stable and unstable regions. A central question concerning the system dynamics is the existence of a critical population N_c for which the phase space consists of a single ergodic and mixing component. If N_c exists, then for greater or equal populations the usual assumptions of ergodicity apply and the equilibrium properties are correctly represented by the microcanonical ensemble of statistical physics.

In order to obtain some insight concerning the system's ergodic properties, in a previous article [1] we investigated the stability of two classes of periodic orbits constructed in the $\mu(x,v)$ space of the one-dimensional (1D) self-gravitating system. The main idea was to examine the stochastic properties of the 1D system and to determine if there is a critical system population above which each type of periodic orbit becomes unstable. By calculating Lyapunov characteristic numbers (LCN's) for small perturbations of the periodic orbits, we concluded that stable orbits exist in 1D systems with populations of 10, demonstrating that the phase space is not everywhere chaotic.

For $N \geq 11$ the LCN's indicated that both classes of periodic orbits became unstable, but that their LCN's apparently differed, suggesting that the phase space was segmented into separate ergodic components. To investigate this apparent segmentation in more detail, the dependence of a given LCN on the "distance" between the reference orbit and the exact periodic orbit was studied. It was observed that in many instances, different perturbations produced distinctly different LCN's, which suggested that the entire phase space is not connected for $N \geq 11$. Segmentation of the phase space was observed through $N = 20$ (the largest system considered) and we concluded that the critical system population that leads to chaos is $N_c > 20$.

A nagging concern regarding these preliminary conclusions was the adequacy of the length of the run times used to obtain the LCN's. It is anticipated that the time required to adequately sample the phase space increases dramatically with the dimension, and hence system population. Although a dynamical evolution time of $50\,000 t_c$ (where t_c is the approximate time for a member to traverse the system) had been used for most computer simulations of the systems examined, we suggested that simulations of much longer duration were needed to ensure convergence to the LCN's found in each case. That is the theme of this research.

New computer hardware and the availability of additional CPU time have enabled us to study the stability of these periodic orbits for significantly longer evolution times. In the current work we have found that for $N \leq 10$ all of the *unstable* trajectories examined appear to converge to a common LCN for a given N. Stable trajectories were still found, validating our previous conclusion that 1D systems of population $N \leq 10$ are not completely chaotic and therefore can never assume the equilibrium distributions derived by Rybicki [2]. For systems with $11 \leq N \leq 30$ all trajectories examined appeared to converge to a common LCN for a given N. However, this convergence is slow and has a conservative lower bound of $31 \times 10^6 t_c$ for $N = 11$. If no other stable orbits exist, the phase space may be completely connected and 1D systems (for $N \geq 11$) are chaotic. Although this conclusion has been reached through extrapolation, the data strongly support a critical population $N_c = 11$ for completely chaotic behavior in 1D systems.

A complete description of 1D systems and the procedures employed in this research are given in our previous article [1]. However, in order to maintain continuity, a brief review is presented here. This is followed by an analysis and discussion of the new results.

II. DESCRIPTION OF THE 1D SYSTEM

The one-dimensional self-gravitating system is composed of N identical mass sheets, each of uniform mass density and infinite in extent in the (y,z) plane. The sheets move freely along the x axis and accelerate as a result of their mutual gravitational attraction. The ith sheet experiences a constant acceleration given by

$$A_i=(2\pi G/N)(N-2i+1) , \qquad (1)$$

where N^{-1} is the mass of a sheet and G is the universal gravitational constant. At an encounter the sheets pass freely through each other. The energy of the system is expressed as

$$E=(1/2N)\sum_{i=1}^{N} v_i^2+(2\pi G/N^2)\sum_{\substack{i,j\\ i<j}} |x_j-x_i| , \qquad (2)$$

where v_i and x_i are the velocity and position of the ith particle, respectively.

The equilibrium velocity and position probability density functions for this system were developed by Rybicki [2]. In the limit that $N\rightarrow\infty$ these functions are

$$\theta(\eta)=\pi^{-1/2}\exp(-\eta^2) \quad (\text{velocity}) , \qquad (3)$$

$$\rho(\xi)=\tfrac{1}{2}\text{sech}^2\xi \quad (\text{position}) , \qquad (4)$$

where

$$\eta=(v/2)(3M/E)^{1/2} \qquad (5)$$

and

$$\xi=(3\pi GM^2/2E)x ; \qquad (6)$$

v, x, M, and E represent the velocity, position, total system mass, and total system energy, respectively.

The dynamical time required for a sheet to traverse the system is referred to as the characteristic time and has been expressed by Luwel, Severne, and Rousseeuw (LSR) [3] in terms of the maximum value of the equilibrium density function $\rho(\eta)$,

$$t_c=(GM\rho_{\max}/\pi)^{-1/2} . \qquad (7)$$

III. MOTIVATION AND HISTORICAL BACKGROUND

The motivational factors for studying the one-dimensional system are numerous. More than four decades ago, Oort [4] and Camm [5] suggested that the system may be an appropriate model for the motion of stars in a direction normal to the disk of a highly flattened galaxy. Thirty years ago, Eldrige and Feix [6] showed the relevance of this system to plasma physics and twenty years ago, Cuperman, Hartman, and Lecar [7] used the system to test conjectures concerning mechanisms for violent relaxation.

Numerical simulations have been performed by a number of investigators in order to study the stochastic properties of 1D systems and the dynamical time required for these systems to progress to an equilibrium distribution or to "thermalize." In 1967 Hohl [8] suggested that 1D systems should relax on the order of N^2t_c. However, research by Wright, Miller, and Stein in 1982 [9] using statistical tests based on the exact velocity and position equilibrium density functions derived by Rybicki suggested that 1D systems do not even approach equilibrium after $2N^2t_c$. A few years later, Luwel, Severne, and Rousseeuw [3] concluded from their investigations that, for a specific class of counterstreamed conditions in the $\mu(x,v)$ space and an initial virial ratio of 0.3, relaxation takes place within Nt_c. In an attempt to resolve the confusion, we [10–13] developed different procedures to test for thermalization and studied the evolution of an assortment of initial states. Relaxation was clearly not found for any of the cases. The initial state suggested by LSR appeared to enter a macrostate that mimics equilibrium and slowly drifted away from it.

For small-N systems ($N\leq 10$) Froeschle and Scheidecker (FS) [14] studied the rate of divergence of 100 orbits and found for $N>5$ no integrable orbits existed and conjectured that these systems are ergodic. This was confirmed by Benettin, Froeschle, and Scheidecker (BFS) [15], who calculated Lyapunov characteristic numbers for $N\leq 10$ and demonstrated an increasing stochasticity with increasing N. The conclusions reached by FS and BFS were later supported by research performed by Wright and Miller (WM) [16]. They studied systems for $N<10$ and found for $N>4$ that relaxation seemed to occur in a time $\gg N^2t_c$ ($\approx 10^5t_c$).

The work of FS, BFS, and WM used randomly generated orbits. By selecting periodic orbits for small N, we found the existence of stable regions in the phase space [1]. FS had suggested the possibility of small integrable regions in the phase space that were too small to be detected by their technique. We also found a segmentation in the phase space for $N\geq 11$. A refinement of this conjecture is found in the data and results section of this paper.

IV. LYAPUNOV CHARACTERISTIC NUMBERS

In a stochastic region of phase space, nearby orbits of dynamical systems diverge exponentially [17]. This exponential divergence may be determined by calculating Lyapunov characteristic numbers [18]. This procedure has been described and used extensively by Contopoulos and Barbanis [19], Contopoulos, Galgani, and Giorgilli [20], and BFS [15]. The Lyapunov characteristic number is defined as

$$L=\lim_{\substack{d_0\rightarrow 0\\ t\rightarrow\infty}}[\ln(d/d_0)/t] , \qquad (8)$$

where d and d_0 are the deviations for two nearby orbits at times t and 0. For stable orbits the Lyapunov characteristic number is zero and positive for unstable orbits. In stochastic regions of the phase space, the use of Eq. (8)

to determine the Lyapunov characteristic number will eventually lead to an overflow. Numerically, this can be avoided if after some time ΔT we rescale d and define the Lyapunov characteristic number as [18]

$$L=\lim_{n\to\infty}(1/n\Delta T)\sum_{j=1}^{n}[\ln(d_{j/d_0})]\ , \tag{9}$$

where $n=t/\Delta T$ and represents the number of rescalings.

The separation in phase space of a particular reference orbit and some nearby orbit is given by

$$d_0=k\left[\sum_{i=1}^{N}[(x_{0ir}-x_{0in})^2+(v_{0ir}-v_{0in})^2]\right]^{1/2} . \tag{10}$$

Here x_{0ir} and v_{0ir} refer to the initial position and velocity, respectively, of the ith sheet of the reference orbit. Similarly x_{0in} and v_{0in} represent the initial conditions of the nearby orbit. The constant k is dependent on N. After some time ΔT the two orbits are separated by

$$d_j=k\left[\sum_{i=1}^{N}[(x_{ir}-x_{in})^2+(v_{ir}-v_{in})^2]\right]^{1/2} . \tag{11}$$

The nearby orbit is then rescaled according to

$$\tilde{x}_{in}=x_{ir}+(d_0/d_j)(x_{in}-x_{ir})\ , \tag{12}$$

$$\tilde{v}_{in}=v_{ir}+(d_0/d_j)(v_{in}-v_{ir})\ , \tag{13}$$

where $\tilde{x}_{in}$ and $\tilde{v}_{in}$ respectively represent the rescaled position and velocity of the ith sheet of the nearby orbit. The evolution of the orbits is then resumed and the reference orbit is considered to be stable if the Lyapunov characteristic number in Eq. (9) coverges to zero and unstable if it converges to some positive number.

V. SELECTED PERIODIC ORBITS

Two simple periodic orbits were chosen in the $\mu(x,v)$ space and are illustrated in Fig. 1. The "breathing mode" is shown in Fig. 1(a) for a five sheet (illustrated as particles) system. Initially, all sheets start from rest at appropriate locations on the x axis. After sometime τ the sheets converge simultaneously at the origin. They pass through each other and after an additional time increment τ the system, except for labeling of the sheets, returns to its initial configuration. Similarly, the periodic orbit which we refer to as "mode 1" is shown in Fig. 1(b) for $N=6$.

VI. SCALING, SIMULATION, AND PERTURBATION

All initial positions and velocities were scaled according to Eqs. (5) and (6) with $M=1$ and $2\pi G=1$. This resulted in a characteristic time of 2π and forced a total dimensionless energy of three-fourths for all systems. The center of mass and total momentum were constrained to zero. The evolution of each system was simulated using an exact code with updating occurring at each encounter. The LCN's were rescaled according to Eqs. (12) and (13) every $0.1t_c$. All calculations were performed in double precision (16 significant figures) on a VAX 4000-500 computer, and energy was conserved to one part in 10^{10}.

All perturbations to the periodic orbits were made such that a sphere of radius r around an initial point in phase space could be examined. For a particular perturbation $P>0$, the perturbed positions and velocities were selected as follows:

$$x_{i\text{perturbed}}=x_{i\text{periodic}}(1\pm P)\ , \tag{14}$$

$$v_{i\text{perturbed}}=v_{i\text{periodic}}(1\pm P)\ . \tag{15}$$

For subsequent sheets the signs of each equation were simply alternated. One sign was not alternated in order to avoid symmetry. The perturbed positions and velocities were then rescaled so that the total energy of the perturbed system was $\frac{3}{4}$. These perturbed orbits were used as the reference orbits for calculation of the LCN's. The reference and nearby orbits were separated by $d_0\approx10^{-10}$.

VII. DATA AND RESULTS

Dynamical time runs of $4.5\times10^5 t_c$ were executed for systems of $N=5$, 6, 10, and 11 sheets. The LCN's shown in Fig. 2 illustrate the perturbations of 14.21% and 15.82% from mode 1 result in stable trajectories. Breathing mode perturbations of 9×10^{-8}%, 0.96%, and 16.31% as well as mode 1 perturbations of 19.07% and 23.97% appear to be converging to a common LCN at about $4\times10^5 t_c$. However, in the same time frame for $N=6$, as shown in Fig. 3, three unstable orbits appear to converge to a common LCN and two do not. A perturbation of 9.75% from mode 1 still results in a stable tra-

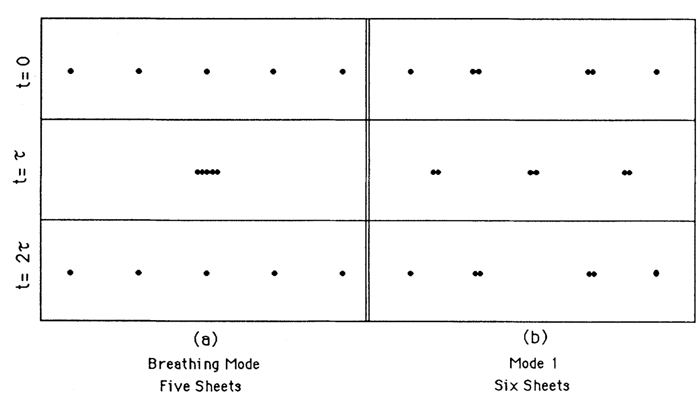

FIG. 1. Illustration of the collision sequence for the breathing mode (B) and mode 1 ($M1$).

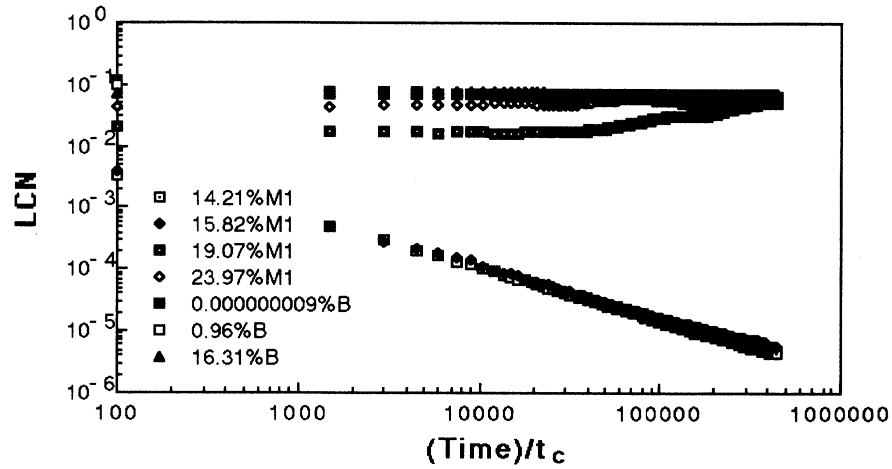

FIG. 2. LCN's for five sheets evolved through $450\,000t_c$ for the breathing mode (B) and mode 1 ($M1$).

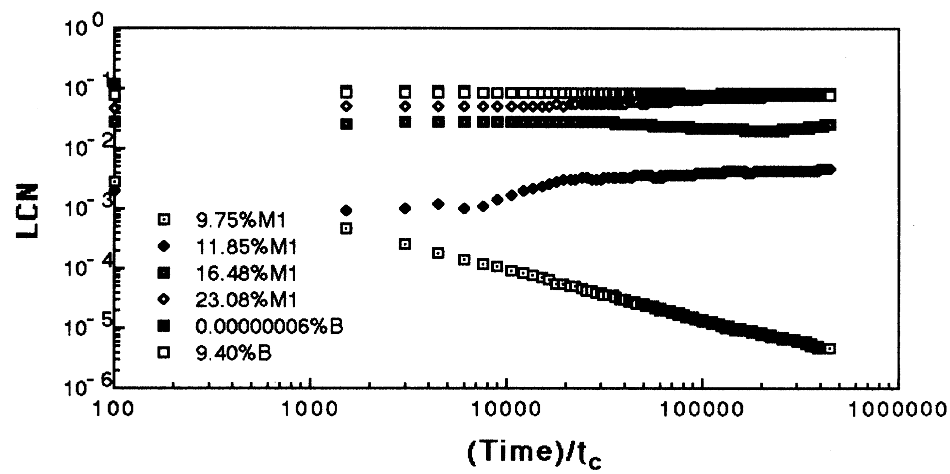

FIG. 3. LCN's for six sheets evolved through $450\,000t_c$ for the breathing mode (B) and mode 1 ($M1$).

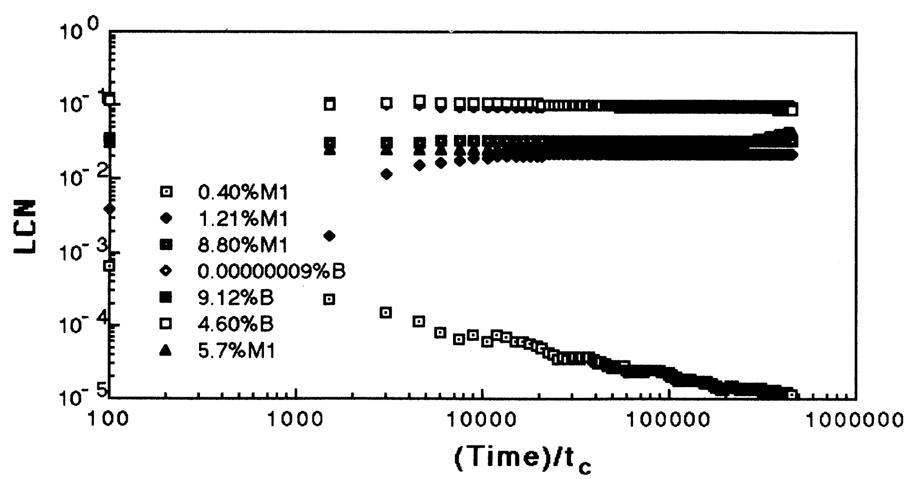

FIG. 4. LCN's for ten sheets evolved through $450\,000t_c$ for the breathing mode (B) and mode 1 ($M1$).

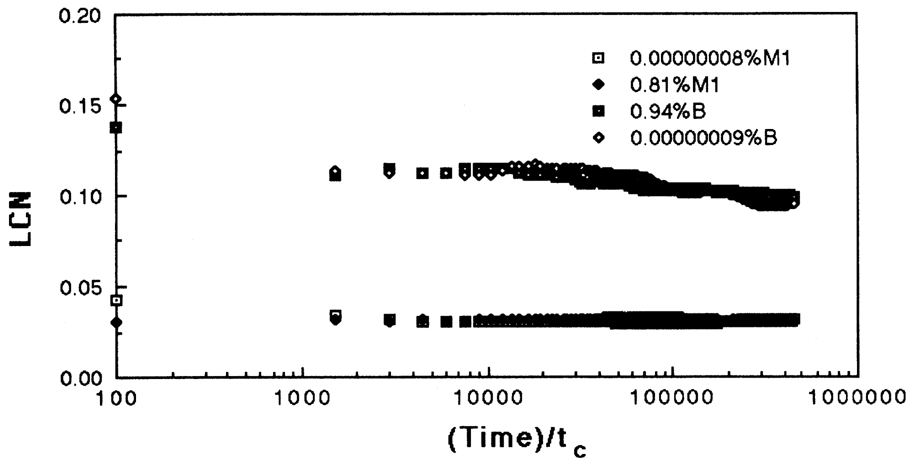

FIG. 5. LCN's for eleven sheets evolved through $450\,000t_c$ for the breathing mode (B) and mode 1 ($M1$).

jectory. Similar results are found in Fig. 4 for $N=10$ with a 0.40% perturbation from mode 1 providing a stable orbit. For $N=11$, Fig. 5 illustrates no stable trajectories were found for perturbations as small as $8\times10^{-8}\%$ from mode 1. This confirms our previous conclusion [1] that 1D systems with $N\leq10$ can never progress to an equilibrium distribution since at least one stable trajectory was found for each system during a simulation time of 4.5×10^5t_c. However, the question remains as to whether or not the phase space is segmented for $N\geq11$. Segmentation precludes thermalization.

Segmentation appears to occur in the unstable trajectories for $N=6$, 10, and 11. However, this segmentation may vanish for longer simulation times as suggested by $N=5$. To check for this possibility, mode 1 perturbations of 9.75%, 11.85%, and 16.48% and a breathing mode perturbation of $6\times10^{-8}\%$ were simulated through 4×10^6t_c for $N=6$. The stable trajectory clearly remains stable and the unstable trajectories begin a trend towards a common LCN at about 2×10^6t_c. This is shown in Figs. 6 and 7. Perhaps a similar convergence is also true for $N=11$. To check for this a mode 1 perturbation of $8\times10^{-8}\%$ and a breathing mode perturbation of $9\times10^{-8}\%$ were permitted to evolve through 15×10^6t_c. A trend for convergence to a common LCN is evident as revealed in Fig. 8.

If a time scale can be found for this convergence, this may represent a thermalization time for the system. To run the system for additional time was out of the question since a single simulation for 15×10^6t_c required approximately 1×10^6 s of CPU. As a crude approximation, we opted for an extrapolation using a least-squares fit of the data from 10×10^6t_c to 15×10^6t_c with the LCN plotted as a function of 1/time. We obtained a result of 31×10^6t_c as shown in Fig. 9.

A short run of 2×10^4t_c was carried out for $N=30$. The same trend was observed as for $N=11$. However, since the data were very limited we can only suggest without much confidence that the time for the trajectories to merge to a common LCN is approximately a factor of 10 greater than that for $N=11$.

VIII. CONCLUSIONS

The extended runs in this research clearly support our previous conclusion that 1D systems with populations $N\leq10$ are not ergodic and therefore thermalization to an equilibrium distribution is not possible for them. Howev-

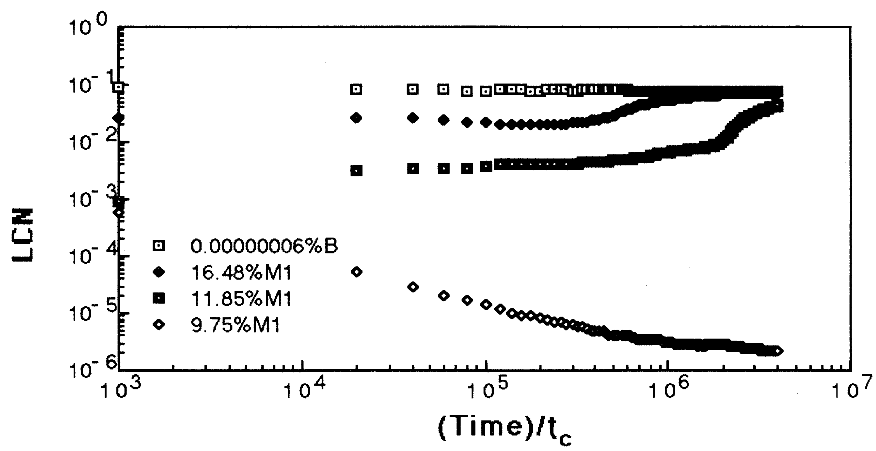

FIG. 6. LCN's for six sheets evolved through $4\,000\,000t_c$ for the breathing mode (B) and mode 1 ($M1$).

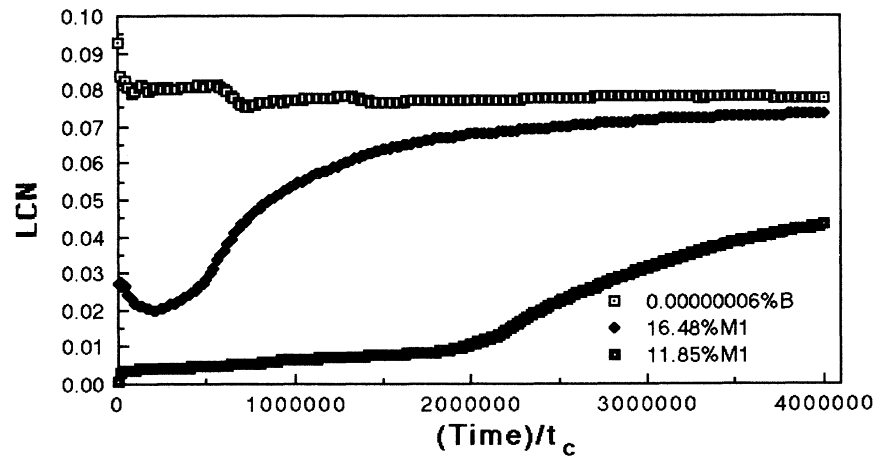

FIG. 7. LCN's for unstable orbits of six sheets evolving through $4\,000\,000t_c$ for the breathing mode (B) and mode 1 ($M1$).

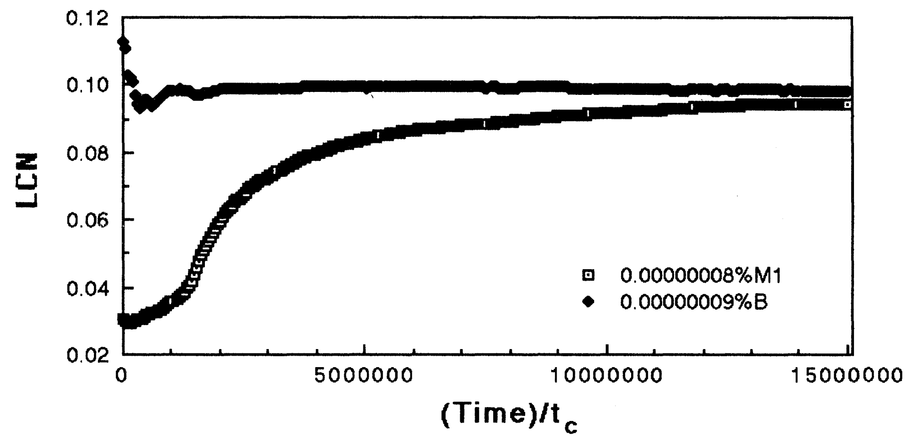

FIG. 8. LCN's for eleven sheets evolved through $15\,000\,000t_c$ for the breathing mode (B) and mode 1 ($M1$).

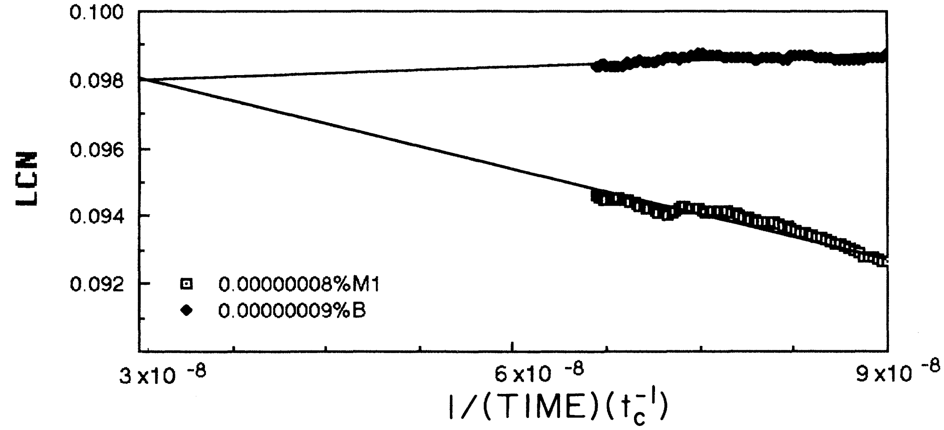

FIG. 9. Extrapolation of LCN convergence time for eleven sheets.

er, contrary to our previous results, the critical population for which 1D systems may become chaotic is apparently $N_c = 11$, and not $N_c > 20$ as reported earlier [1]. Of course, this conjecture is based on two assumptions: First, no stable trajectories exist for $N \geq 11$, and second, no other periodic trajectory is situated in an isolated phase space segment. It is supported by the apparent convergence of LCN's initiated near distinct periodic trajectories to the same value.

It is unlikely that there is a stable periodic trajectory for $N \geq 11$ since periodic orbits require, at a minimum, pairs of simultaneous encounters. WM [16] studied the relationship between the encounter sequence and the rate of divergence of proximally initialized pairs of trajectories. They found that trajectories which contain nearly multiple encounters diverge rapidly and the proportion of these encounters increases with N. We have also analytically searched for a periodic orbit where, at most, a single encounter occurs at a given time, but to date none has been found.

We have no direct way of knowing if there are other periodic trajectories associated with a further segmentation of the phase space. We do note, however, that several of the perturbations were quite large and an apparent convergence to a common LCN was seen for each. This covers a wide range of possible orbits.

All of this evidence points to the conclusion that 1D systems are chaotic for $N \geq 11$. A possible thermalization time is $31 \times 10^6 t_c$ for $N = 11$, but this is only a rough lower-bound estimate. We also suggest, with very little confidence, that the thermalization time for $N = 30$ has a lower bound $\approx 3 \times 10^8 t_c$. If this conjecture is correct then there may be a rapid increase in the thermalization time with an increase in the system population. Evidently, an increase in the number of degrees of freedom restricts, in a fashion similar to Arnold's diffusion [21], the rate at which a small perturbation from an unstable, periodic orbit explores the phase space.

Although this research gives no accurate estimation of the thermalization times for 1D systems, we are quite confident that these times are excruciatingly long and explains why to date no system has been observed to thermalize. It is unlikely that the thermalization of systems with $N > 100$ will be seen with present computer technology.

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